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Singularities of minima: a walk on the wild side of the Calculus of Variations

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Abstract I will report on some recent developments concerning the problem of estimating the Hausdorff dimension of the singular sets of solutions to elliptic and variational problems. Emphasis will be given on some open issues. Connections with measure data problems will be outlined.

Keywords Partial regularity · Singular sets · Fractional differentiability

1 Wild sets

An important feature of elliptic and variational problems in the vectorial case is the one of exhibiting singularities: while under reasonable assumptions solutions are everywhere regular in the scalar case, in the vectorial one they turn out to be continuous only outside a negligible, closed subset, called *the singular set*, which can be a very wild one. It is the aim of this note to point out some recent developments in the study of singular sets, some connections to other regularity issues, and a few open problems. To begin with, let me consider variational integrals of the type

$$\mathcal{F}(v, A) := \int_A F(x, v, Dv)\, dx, \tag{1}$$

defined for Sobolev maps $v \in W^{1,p}_{\rm loc}(\Omega, \mathbb{R}^N)$, and open sets $A \subseteq \Omega$; I shall often denote $\mathcal{F} \equiv \mathcal{F}(v) \equiv \mathcal{F}(v, \Omega)$. Here $n \geq 2$, $N \geq 1$, Ω is a bounded open set in $\mathbb{R}^n$, while, unless otherwise specified, I am assuming $p > 1$. The integrand $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is of

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class C^2 in the third variable, satisfying

$$\begin{cases} \nu|z|^p \le F(x,v,z) \le L(1+|z|^p) \\ \nu(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \le \langle F_{zz}(x,v,z)\lambda,\lambda\rangle \le L(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \\ |F(x,u,z)-F(y,v,z)| \le L\omega(|x-y|+|u-v|)(1+|z|^p)\ , \end{cases} \tag{2}$$

for all $x, y \in \Omega, u, v \in \mathbb{R}^N$ and $z, \lambda \in \mathbb{R}^{N\times n}$, where $0 < \nu \le L$ and $\omega\colon \mathbb{R}^+ \to (0,1)$ is a continuous, non-decreasing modulus of continuity, such that for some $\alpha \in (0,1)$,

$$\omega(s) \le s^\alpha. \tag{3}$$

Assumption $(2)_2$ describes a controlled, uniform convexity of the integrand F, via growth conditions imposed on the second derivatives F_{zz}, which are in turn prescribed accordingly to the ones in $(2)_1$. Convexity in the gradient variable of F is the main assumption considered above. Assumption $(2)_3$, together with (3), means that $(x,v) \mapsto F(x,v,z)/(1+|z|)^p$ is Hölder continuous with respect to (x,v) with exponent $\alpha \in (0,1)$, uniformly with respect to z.

Local minimizers of $\mathcal{F}$ are maps $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ such that $\mathcal{F}(u,A) \le \mathcal{F}(v,A)$, whenever $A \subset\subset \Omega$ and $u - v \in W_0^{1,p}(A,\mathbb{R}^N)$.

Beside functionals I will consider homogeneous non-linear elliptic systems in divergence form

$$\operatorname{div} a(x,u,Du) = 0\ , \tag{4}$$

where $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is a continuous vector field of class C^2 in the third variable, satisfying

$$\begin{cases} |a(x,v,z)| \le L(1+|z|^{p-1}) \\ \nu(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \le \langle a_z(x,v,z)\lambda,\lambda\rangle \le L(1+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \\ |a(x,u,z)-a(y,v,z)| \le L\omega(|x-y|+|u-v|)(1+|z|^{p-1})\ , \end{cases} \tag{5}$$

for all $x, y \in \Omega, u, v \in \mathbb{R}^N$ and $z, \lambda \in \mathbb{R}^{N\times n}$, and $\omega\colon \mathbb{R}^+ \to (0,1)$ is as in (3). A weak solution to (4) is of course a map $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ such that

$$\int_\Omega a(x,u,Du)D\varphi\,dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega)\ . \tag{6}$$

In the following, when using the word regular for a map I will usually mean "Hölder continuous" or, more often, "with Hölder continuous gradient". The latter is the focal point of the regularity theory, after which higher regularity of solutions can be obtained via standard boot-strap arguments.

In the *scalar case* $N = 1$, when minima and solutions are real-valued functions, singularities do not show up and minimizers are everywhere regular in the interior:

Theorem 1 *Let $u \in W^{1,p}(\Omega)$ be either a local minimizer of the functional $\mathcal{F}$ under the assumptions (2)–(3), or a solution to (4) under the assumptions (5)–(3). Then $Du \in C^{0,\alpha/2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ in the case of functionals, and $Du \in C^{0,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ in the case of equations.*

See for instance [30,45]. The main ingredient here is a suitable application of ideas going back to DeGiorgi [15], and these are essentially scalar ones. In terms of pointwise regularity of solutions, the last two results are the best possible. Notice the decrease in the Hölder exponent as we move from solutions to minima: counterexamples show that minimizers are not $C^{0,\alpha}$-regular in general, they are only $C^{0,\alpha/2}$-regular.

In the *vectorial case* $N > 1$ the situation drastically changes and minima and solutions generally lose a lot of regularity properties. This was first shown by DeGiorgi [16]; then came a series of counterexamples, including the one of Nečas [53] who showed that even in the case of an analytic integrand of the type $F(x, u, Du) \equiv F(Du)$ minima to (1) may be not $C^{1,\alpha}$ for any α. The minimizer found by Nečas was Lipschitz continuous: the possibility of lowering the regularity of the counterexamples remained an open problem. This was settled by Šverák and Yan [57], who showed that even when considering the favorable case of smooth and quadratic-growth functionals as Nečas, there exist unbounded minimizers. In general the singular set in non-empty, but when considering very special structures [36,58].

After realizing that solutions to vectorial problems can be irregular the next step is the so-called partially regularity, i.e. *proving that the regularity properties of the scalar case are preserved at least on large subsets.* Typical partial regularity statements are

Theorem 2 *Let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of the functional* $\mathcal{F}$*, under the assumptions* (2)–(3)*. Then there exists an open subset* $\Omega_u \subset \Omega$ *such that* $|\Omega \setminus \Omega_u| = 0$*, and* $Du \in C^{0,\alpha/2}_{\rm loc}(\Omega_u, \mathbb{R}^{N\times n})$*.*

Theorem 3 *Let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a weak solution to the system* (4)*, under the assumptions* (5) *and* (3)*. Then there exists an open subset* $\Omega_u \subset \Omega$ *such that* $|\Omega \setminus \Omega_u| = 0$*, and* $Du \in C^{0,\alpha}_{\rm loc}(\Omega_u, \mathbb{R}^{N\times n})$*.*

Theorems 2 and 3 are the vectorial counterpart of the scalar Theorem 1; here we see that Hölder continuity is preserved only on an open subset $\Omega_u \subset \Omega$; the set

$$\Sigma_u := \Omega \setminus \Omega_u \tag{7}$$

is called the singular set of u. The proofs of Theorems 2 and 3 automatically provide a characterization of Σ_u as the set of "non-Lebesgue points" of Du in the sense that

$$\Sigma_u = \Sigma_u^0 \cup \Sigma_u^1 \ , \tag{8}$$

where

$$\Sigma_u^0 := \left\{ x \in \Omega : \liminf_{\rho \searrow 0} \fint_{B(x,\rho)} |Du(y) - (Du)_{x,\rho}|^p \, dy > 0 \text{ or } \limsup_{\rho \searrow 0} |(Du)_{x,\rho}| = \infty \right\}$$

$$\Sigma_u^1 := \left\{ x \in \Omega : \liminf_{\rho \searrow 0} \fint_{B(x,\rho)} |u(y) - (u)_{x,\rho}|^p \, dy > 0 \text{ or } \limsup_{\rho \searrow 0} |(u)_{x,\rho}| = \infty \right\} .$$

Here $|(Du)_{x_0,r}|$, $|(u)_{x_0,r}|$ denote the average of Du, u over the ball $B(x_0, r)$. Indeed the proofs are based on the following fact: a point $x_0 \in \partial\Omega$ is regular, that is Du is Hölder continuous in a neighborhood of x_0, if and only if

$$\fint_{B(x_0,r)} |Du(x) - (Du)_{x_0,r}|^p \, dx \le \varepsilon \ , \tag{9}$$

where $\varepsilon > 0$ is suitably small number; moreover $|(Du)_{x_0,r}|$ must stay bounded at every scale r. The proofs also require a similar condition for u. Since such conditions are satisfied almost everywhere by Lebesgue's differentiation theory, partial regularity with the characterization in (8) follow.

In the form given here Theorems 2 and 3 are the result of the efforts of several authors, and their first versions were originally and independently obtained by Giaquinta, Giusti and Modica [26–28], and independently by Ivert [37,38]; the methods of such authors partially rely on suitable freezing techniques, originally pioneered by Campanato [8] in the case of linear elliptic equations. The use of Campanato spaces is indeed fundamental here. For the present optimal form I refer to Duzaar and Grotowski [18], who introduced an interesting new technique coming from Geometric Measure Theory problems: the A-harmonic approximation method [20].

Now a natural question arises: *how large can the singular set* Σ_u *be?* The problem is the following: let $\dim_{\mathcal{H}}(\Sigma_u)$ denote the Hausdorff dimension of Σ_u, then

$$\text{“Is } \dim_{\mathcal{H}}(\Sigma_u) < n \text{ true”?} \tag{10}$$

Question (10) is the most natural one after proving partial regularity. The first answers, dating back to the seventies, are for the special case $\operatorname{div} a(Du) = 0$, when

$$\dim_{\mathcal{H}}(\Sigma_u) \leq n-2\ . \tag{11}$$

The latter estimate is essentially obtained using the classical results asserting the existence of second derivatives of solutions, that in turn implies (11); see [10,22]. The validity of (11) has been proved by Campanato and Cannarsa [9] for solutions to certain higher order systems, in cases to which formerly existent techniques did not apply. Such results immediately apply to functionals as in (1) when $F(x, u, Du) \equiv F(Du)$, via the use of the Euler-Lagrange system. A first progress toward general structures was made in [26] that treated the very special case $F(x, u, Du) \equiv c(x, u)|Du|^p$.

Question (10) for functionals (1) and systems (4) was raised immediately after the first proofs of Theorems 2 and 3, both for systems - see [22], page 191, and [28] page 115 – and for functionals – [23] Sect. 3, and [25] open problem (a), page 117.

The general answer to (10) is "yes" as shown in [42–44,46,47]. In the next section I shall present an approach to the problem, together with estimates for the Hausdorff dimension of the singular sets.

2 L^p-Hölder continuity and singular sets

The approach I will follow here is to outline a duality between the loss of $C^{0,\alpha}$-regularity of Du and presence of the singular set Σ_u on one side, and persistance of a weaker form of Hölder continuity and estimation of $\dim_{\mathcal{H}}(\Sigma_u)$ on the other one.

A measurable map $w : \Omega \to \mathbb{R}^N$ belongs to the *fractional Sobolev space* $W^{\sigma,p}(\Omega, \mathbb{R}^N)$ for parameters $\sigma \in (0, 1)$ and $p \in [1, \infty)$, provided the following Gagliardo-type norm is finite:

$$\|w\|_{W^{\sigma,p}(\Omega)} = \left(\int_\Omega |w(x)|^p\, dx\right)^{\frac{1}{p}} + \left(\int_\Omega \int_\Omega \frac{|w(x)-w(y)|^p}{|x-y|^{n+\sigma p}}\, dx\, dy\right)^{\frac{1}{p}}. \tag{12}$$

The map w belongs to the Nikolski space $\mathcal{N}^{\sigma,p}(\Omega, \mathbb{R}^N)$ if and only if

$$\|w\|_{\mathcal{N}^{\sigma,q}(\Omega)} := \|w\|_{L^p(\Omega)} + \sup_{h,e}\left(\int_{A_h} \frac{|w(x+he)-w(x)|^p}{|h|^{\sigma p}}\right)^{\frac{1}{p}}, \tag{13}$$

where the supremum is taken considering $h \in R \setminus \{0\}$, $e \in R^n$ such that $|e| = 1$, and $A_h := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > |h|\}$. The local variants $W^{\sigma,p}_{\mathrm{loc}}(\Omega)$ and $\mathcal{N}^{\sigma,p}_{\mathrm{loc}}(\Omega)$ are defined in the obvious way, and the strict inclusions

$$W^{\sigma,p}(\Omega) \subset \mathcal{N}^{\sigma,p}(\Omega) \subset W^{\sigma-\varepsilon,p}(\Omega) \tag{14}$$

hold for every $\varepsilon \in (0, \sigma)$. Such spaces are particular instances of another family of spaces, the so called Besov spaces $B^{\sigma}_{p,q}$ [55]: indeed $B^{\sigma}_{p,p} \equiv W^{\sigma,p}$, $B^{\sigma}_{p,\infty} \equiv \mathcal{N}^{\sigma,p}$, and $B^{\sigma}_{\infty,\infty} \equiv C^{0,\sigma}$.

Now, let us read $C^{0,\sigma}$-regularity as $\sigma-$Hölder continuity in the L^∞-norm

$$\|Du(x+h) - Du(x)\|_{L^\infty(\Omega')} \leq [Du]_{0,\sigma}|h|^\sigma \qquad \Omega' \subset\subset \Omega\,, \tag{15}$$

for any $h \in \mathbb{R}^n$ such that $|h| \leq \mathrm{dist}(\Omega', \partial\Omega)$. Whilst (15) is in general lost in the vectorial case due to the presence of the singular set, it happens that the gradient belongs to some Nikolski space, and so Hölder continuity is kept in a weaker norm

$$\|Du(x+h) - Du(x)\|_{L^p(\Omega')} \leq \|u\|_{\mathcal{N}^{\sigma,q}(\Omega)}|h|^\sigma \quad \Omega' \subset\subset \Omega, \quad 1 < p < \infty\,. \tag{16}$$

In other words *when passing from the scalar to the vectorial case Hölder continuity is preserved modulo being read in the right form*. The natural dual aspect of this fact is that an estimate like (16) allows one to bound the Hausdorff dimension of the singular set: indeed, see for instance [46], a classical result in potential theory is

Theorem 4 *Let $w \in W^{\sigma,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^k)$, where $\sigma \in (0,1]$, $p \geq 1$ are such that $\sigma p < n$. Let Σ_u denote the set of non-Lebesgue points of w in the sense of*

$$\Sigma_w := \left\{ x \in \Omega : \liminf_{\rho \searrow 0} \fint_{B(x,\rho)} |w(y) - (w)_{x,\rho}|^p \, dy > 0 \text{ or } \limsup_{\rho\searrow 0} |(w)_{x,\rho}| = \infty \right\}.$$

Then its Hausdorff dimension $\dim(\Sigma_w)$ *satisfies* $\dim(\Sigma_w) \leq n - \sigma p$.

The strategy is now clear: ones uses (16) to bound the Hausdorff dimension of singular set via Theorem 4 and the characterization (8); keeping (14) in mind. We shall now report on some results. The first I report is for special structures, and is taken from [43].

Theorem 5 *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional*

$$\mathcal{F}(v) \equiv \int_\Omega f(x, Dv) + g(x, v)\, dx,$$

under the assumptions (2),(3). *Then*

$$Du \in W^{\frac{\alpha-\varepsilon}{p},p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}), \quad \forall \varepsilon \in (0, \alpha). \tag{17}$$

Therefore, denoting by $\Sigma_u \subset \Omega$ the singular set of u in the sense of (7)*, we have* $\dim_{\mathcal{H}}(\Sigma_u) \leq n - \alpha$.

For a special class of systems I quote the following result from [46]:

Theorem 6 *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the system*

$$\mathrm{div}\, a(x, Du) = 0\,, \tag{18}$$

under the assumptions (5),(3). *Then*

$$Du \in W^{\frac{2\alpha-\varepsilon}{p},p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}), \quad \forall \varepsilon \in (0, 2\alpha). \tag{19}$$

Therefore, denoting by $\Sigma_u \subset \Omega$ *the singular set of u in the sense of* (7)*, we have* $\dim_{\mathcal{H}}(\Sigma_u) \leq n - 2\alpha$.

Notice the analogy between the estimate of Theorem 6 and (11): the latter is recovered for Lipschitz continuous coefficients (Theorem 6 actually applies when $\alpha = 1$ too, see [46] again). Theorems 5–6 are "twins", and should be compared to Theorems 2–3: in both cases the rate of Hölder continuity of the coefficients (3) influences that of the solutions in the similar way, but we pass from the $C^{0,\alpha}$-regularity of the scalar case, to the α-Hölder continuity in the L^p-sense of (17)–(19); again keeping (14) in mind. In turn the regularity of the coefficients reflects on the dimension estimates for the singular sets. *It would be very interesting to discuss the optimality of the singular set estimates in the last two theorems.* At the moment I have no conjectures on this.

When passing to the general structures in (1) and (4) things considerably worsen, and I first have to recall that both in the case of minimizers of the functional (1), and of solutions to (4), so-called higher integrability holds: there exists a number q, depending only on n, N, p and L/ν, such that

$$Du \in L^q_{\mathrm{loc}}(\Omega), \quad q > p. \tag{20}$$

The dependence on p is harmless: assume $p \in [\gamma_1, \gamma_2] \subset (1, \infty)$; then q only depends on γ_1, γ_2. Inclusion (20) holds even under the only assumption $(2)_1$ for functionals, and $(5)_2$ for systems, and it is basically a consequence of the celebrated Gehring's lemma [39].

With the former higher integrability property of Du available, we have the following result from [43]:

Theorem 7 *Let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a local minimizer of the functional* $\mathcal{F}$ *under the assumptions* (2)–(3)*. Then*

$$Du \in W^{\frac{\sigma_0 - \varepsilon}{p}, p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}), \qquad \sigma_0 := \min\{\alpha, q - p\} \qquad \forall\, \varepsilon \in (0, \sigma_0). \tag{21}$$

Therefore, denoting by $\Sigma_u \subset \Omega$ *the singular set of u in the sense of* (7)*, we have* $\dim_{\mathcal{H}}(\Sigma_u) \leq n - \sigma_0 < n$.

Instead, for the case of systems the following theorem holds [47]:

Theorem 8 *Let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a weak solution to the system* (4) *under the assumptions* (5),(3)*. Then*

$$Du \in W^{\frac{\sigma_1 - \varepsilon}{p}, p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n}), \qquad \sigma_1 := \min\{2\alpha, q - p\} \qquad \forall\, \varepsilon \in (0, \sigma_1)\,. \tag{22}$$

Therefore, denoting by $\Sigma_u \subset \Omega$ *the singular set of u in the sense of* (7)*, we have* $\dim_{\mathcal{H}}(\Sigma_u) \leq n - \sigma_1 < n$.

In the above-mentioned papers [47] and [43] the proofs are actually given for the case $p \geq 2$; the case $1 < p < 2$ can be treated with some technical modifications; see for instance [19]. Note that since the difference $q - p$ essentially depends on L/ν, then the bound on the singular set depends essentially on this number via (21)–(22): the integrability properties of the gradient (20) control the size of the singular set. This is explained as follows. Looking at Theorems 5–6 we see that the regularity of the coefficients (3) influences the dimension estimates for the singular set: the higher is α, the smaller $\dim_{\mathcal{H}}(\Sigma_u)$ is ensured to be. When passing to the general structures (1) and (4) then $u(x)$ comes into the play, acting as a measurable, very irregular coefficient; nevertheless (20) can be used to control the oscillations of $u(x)$, and a singular set estimate is still possible, thereby depending on q.

In the same spirit, when $n \le p + 2$ the map $u(x)$ is more regular via the use of certain low dimensional regularization techniques, and therefore better results can be proved. More precisely, we have once again $\dim_{\mathcal{H}}(\Sigma_u) \le n - 2\alpha$ for systems, and $\dim_{\mathcal{H}}(\Sigma_u) \le n - \alpha$ in the case of functionals. As a matter of fact the singular set is always empty in the two-dimensional case $n = 2$; see last section of [43]. Note that the case of functionals poses remarkable additional difficulties with respect to the one of systems. Indeed, while systems *can be tested* by test functions, this is not the case for functionals. Under the assumptions considered here the functional $\mathcal{F}$ is not differentiable, and does not possess the associated Euler-Lagrange system, due to the fact that Hölder continuity u is all is assumed in $(2)_3$. But even in the favorable case that $F(x, u, z)$ is smooth, the Euler-Lagrange system would turn out to be a *system with a right hand side with critical growth*, un-treatable without an L^∞-smallness condition on the minimizer [36]. On the other hand minimizers of smooth functionals are in general unbounded in the vectorial case [57]. The approach adopted in [43] is variational, and essentially uses *the minimality of the map* u. The technique relies on a localization method that makes once again essential use of the Euler-Lagrange systems of certain convex comparison functionals, and finally leads to establish a higher (fractional) differentiability of Du via a comparison method: here minimality is crucial. The situation here is in some sense similar to the one of harmonic maps: critical points of the associated Euler-Lagrange system, that is weakly harmonic maps, can be irregular on a dense set, as shown by Riviére [54], but for minimizers a partial regularity theory is available, as first shown by Schoen and Uhlenbeck [56]. Also for minimizers of functionals as in (1) *crucial information is lost when not using minimality*, that is: just using the fact that a map is a critical point without using its minimality too does not yield regularity information. This is something we are going to see again, and in the most dramatic way, when dealing with quasiconvexity in the next section.

3 Quasiconvexity, potential theory, and set porosity

Convexity is suitable to ensure lower semicontinuity for variational integrals, and therefore existence of minima. In the vectorial case there is another condition, much weaker that convexity, which is sufficient for lower semicontinuity, and actually necessary under certain natural assumptions: this is the so-called quasiconvexity. For the sake of simplicity I shall consider here functionals of the type

$$\mathcal{F}(v) := \int_\Omega F(Dv)\, dx. \tag{23}$$

Introduced by Morrey in [50], quasiconvexity of the integrand $F(\cdot)$ means that

$$\int_{(0,1)^n} [F(z_0 + D\varphi(x)) - F(z_0)]\, dx \ge 0, \tag{24}$$

for every $\varphi \in C_c^\infty((0,1)^n, \mathbb{R}^N)$, and for all $z_0 \in \mathbb{R}^{N\times n}$. Quasiconvexity ensures lower semicontinuity in the weak topology of appropriate Sobolev spaces [1,50]. Though the quasiconvexity of $F(\cdot)$ may seem to depend on the choice of the integration domain in (24), a simple covering argument shows that it is possible to replace $(0, 1)^n$ by any other open subset; more information is for instance in the book of Dacorogna [14]. Quasiconvexity also plays an important role in the theory of non-linear elasticity, and in mathematical materials science, see the papers of Ball [4] and Müller [51]. It is a difficult notion to deal with, basically due to its purely non-local character [41].

Partial regularity as in Theorem 1 holds for minima of quasiconvex integrals, under suitable assumptions. More precisely, condition (24) must be reinforced in the so called uniform-strict quasiconvexity, firstly introduced by Evans [21]

$$\nu \int_{(0,1)^n} (1 + |z_0|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \, dx \le \int_{(0,1)^n} [F(z_0 + D\varphi) - F(z_0)] \, dx \, . \quad (25)$$

Roughly speaking, this plays the same role that $(2)_2$ plays for convex integrals, as the following statement demonstrate [2,11].

Theorem 9 *Let $u \in W^{1,p}(\Omega, R^N)$ be a local minimizer of the functional $\mathcal{F}$ in (23), such that $F(\cdot)$ is a C^2-function satisfying $(2)_1$ and (25). Then there exists an open subset $\Omega_u \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$, and $Du \in C^{0,\alpha}_{\rm loc}(\Omega_u, \mathbb{R}^{N\times n})$, for any $\alpha \in (0, 1)$.*

Also in this case the characterization in (8) holds. The first partial regularity result for minimizers of quasiconvex integrals has been proved in a by now classical paper of Evans [21]. The proof under the essentially optimal conditions considered here is given in [2] for the case $p \ge 2$, and in [11] for $1 < p < 2$. Note that the case $p \in (1, 2)$ poses a few non-trivial technical difficulties, since quasiconvexity does not allow for those duality methods typical of the convex case [33]. The result extends also to general functionals of the form (1), under assumption $(2)_3$.

Again, the first natural question is (10); see [24], Sect. 4.2. *Under the generality of Theorem 9 the problem is still open.* The point is that quasiconvexity is a very delicate and "unstable" notion that prevents the application of many standard convexity techniques. For instance, any approach based on the Euler-Lagrange system seems to require convexity, or some variations of it [34], and the Euler-Lagrange system in itself cannot yield regularity results. This was recently shown by Müller and Šverák [52]: they demonstrated even the absence of partial regularity for critical, non-minimizing points of uniformly strictly quasiconvex integrals of the type in (23). This is not the case for convex functionals, by Theorem 3 applied to the Euler-Lagrange system.

Here I want to describe the first answer to (10), given in [44]. It is restricted to the case of Lipschitz minimizers, a condition anyway satisfied by minimizers in several instances of quasiconvex functionals [12,40].

Theorem 10 *Let $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ in (23), such that $F(\cdot)$ is a C^2-function satisfying $(2)_1$ and (25) with $p \ge 2$, and let $\Sigma_u := \Omega \setminus \Omega_u$ be the singular set of u, in the sense of Theorem 9. Then there exists a positive*

$$\delta \equiv \delta(n, N, p, L/\nu, \|u\|_{W^{1,\infty}}) > 0, \quad (26)$$

also depending on the integrand $F(\cdot)$, but independent of the minimizer u, such that $\dim_{\mathcal{H}}(\Sigma_u) \le n - \delta < n$.

The number δ appearing in (26) is in principle explicitly computable by carefully keeping track of the constants involved in the proof; moreover it stays bounded away from zero as soon as p varies in a compact interval of $[2, \infty)$. It depends on the integrand $F(\cdot)$ via two features only: first, the modulus of continuity of its second derivatives F_{zz}, second, the associated "growth function" $G(M) := \sup_{|z|\le M} |F_{zz}(z)|/(1 + |z|)^{p-2}$. Theorem 10 extends to more general quasiconvex functionals of the type in (1), once again assuming $(2)_3$; interestingly, in this case, contrary of what happened for Theorem 7, δ is still independent of the Hölder continuity exponent α in (3): this is basically the effect of the Lipschitz continuity of the minimizer.

Let me give you a glimpse of the proof. From now on I shall restrict for the simplicity to the quadratic growth case $p = 2$, that already contains many of the essential features of the problem; accordingly I will assume that second derivatives of $F(\cdot)$ are bounded: $|F_{zz}(z)| \leq L$. To prove Theorem 10 Kristensen and I employed a suitably localized form of certain potential theory estimates due to Dorronsoro [17]. In turn this and a suitable Cacciop-poli's type inequality for minima [21] imply that for all balls $B(x_0, 4R) \subset \Omega$ with radius R smaller than a suitable one R_0, it holds

$$\int_{B(x_0,R)} \int_0^R ⨍_{B(x,r)} |Du(y) - (Du)_{x,r}|^2 \, dy \, \frac{dr}{r} \, dx \leq c \int_{B(x_0,2R)} |Du|^2 \, dx. \tag{27}$$

This is Carleson type estimate for the "excess functional"

$$E(x, R) := ⨍_{B(x,r)} |Du(y) - (Du)_{x,r}|^2 \, dy, \tag{28}$$

and it is a first regularity information on the oscillations of Du. I would like to remark here that inequality (27) does not require the Lipschitz continuity of minimizers yet. Now assume that the minimizer u is Lipschitz continuous; then (27) turns to

$$⨍_{B(x_0,R)} \int_0^R ⨍_{B(x,r)} |Du(y) - (Du)_{x,r}|^2 \, dy \, \frac{dr}{r} \, dx \leq c \|u\|^2_{W^{1,\infty}}. \tag{29}$$

The latter inequality allows to prove a geometric property of the singular set called "set porosity", asserting that Σ_u has "holes of uniform size at any scale". More precisely, for every point $x_0 \in \Sigma_u$ and every ball $B(x_0, R) \subset\subset \Omega$, there exists at least another ball $B_{\lambda R} \subset B(x_0, R)$, such that $\Sigma_u \cap B_{\lambda R} = \emptyset$. Here $\lambda \in (0, 1/2)$ essentially depends on $n, N, L/\nu$ and $\|u\|_{W^{1,\infty}}$, but is independent of any of the balls considered. $B_{\lambda R}$ is "the hole", and its size λR is uniform in that λ does not depend on R. In turn, this fact and a standard covering argument allow to prove that $\dim_{\mathcal{H}}(\Sigma_u) \leq n - \delta$, where δ depends on λ, n, and therefore ultimately on $n, N, L/\nu$ and $\|u\|_{W^{1,\infty}}$.

The technique just outlined is completely different from the ones adopted in the convex case, but surprisingly, at the end it shows an interesting connection between the two cases. Indeed, when $F(\cdot)$ is convex we have $Du \in W^{\sigma,2}$ for some $\sigma > 0$, see Theorem 7 and recall that now $p = 2$. In turn a result of potential theory, see [3] Sect. 4.8, yields

$$\int_{B(x_0,R)} \int_0^R \left(⨍_{B(x,r)} \frac{|Du(y) - (Du)_{x,r}|}{r^\sigma} \, dy \right)^2 \frac{dr}{r} \, dx \leq c \|Du\|^2_{W^{\sigma,2}}, \tag{30}$$

for any $B(x_0, 4R) \subset\subset \Omega$. Using now a well known reverse Hölder-type inequality for the excess $E(x_0, R)$ (see [30], inequality (9.54)) it is easy to conclude with

$$\int_{B(x_0,R)} \int_0^R ⨍_{B(x,r)} \frac{|Du(y) - (Du)_{x,r}|^2}{r^{2\sigma}} \, dy \, \frac{dr}{r} \, dx \leq c \|Du\|^2_{W^{\sigma,2}}. \tag{31}$$

The previous inequality is valid provided $\sigma > 0$, and therefore holds only in the convex case. Now (27), which is valid for general, even non-Lipschitzian minima, looks exactly as the borderline case of (31) when $\sigma \searrow 0$, and it is in some the trace of the (fractional) differentiability properties of the gradient that eventually get lost in the passage from convexity to quasiconvexity.

4 A problem on the Dirichlet problem

In this section I want to briefly describe a basic and only partially solved boundary regularity problem. Let me consider the following Dirichlet problem

$$\begin{cases} \operatorname{div} a(x, u, Du) = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \tag{32}$$

where $u_0 \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ and $\partial\Omega$ is $C^{1,\alpha}$-regular, and the assumptions (5)–(3) are in force. The natural question is whether partial regularity extends up-to-the-boundary: can we say for instance that a.e. boundary point, with respect to the usual surface measure, is a regular one? That is: Du is Hölder continuous in a neighborhood (of the relative topology of $\overline{\Omega}$) of the considered boundary point. In fact [31,35] a boundary point $x_0 \in \partial\Omega$ is regular if and only if for some small positive number ε we have

$$⨍_{B(x_0,R)\cap\Omega} |Du - (Du)_{B(x_0,R)\cap\Omega}|^p \, dx < \varepsilon. \tag{33}$$

This is the natural boundary version of the standard interior "ε-regularity" criterium (9). Now, while (9) yields information in the interior case, unfortunately condition (33) does not, since it is a priori verified only a.e. with respect to the Lebesgue measure, while the boundary $\partial\Omega$ is a null set. The problem of finding the existence *of even one regular boundary point* remained totally open even for basic structures as in (18), see comments at page 246 of [22]. The only known answer was available for quasilinear systems $\operatorname{div}(b(u)Du) = 0$, or for regularity of u rather than Du [13,29,32]. This gap is in sharp contrast to what happens in the case of elliptic equations, where full regularity carries up to the boundary. The first general results concerning the existence of regular boundary points for Dirichlet problems as in (32) have been given by Duzaar and Kristensen and I in [19]. The idea is to carry out the estimate of Theorem 5 up to the boundary; then assuming α large implies that a.e. boundary point is regular. When in low dimensions $n \leq p + 2$ larger classes of systems can be considered.

Theorem 11 *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to (32), under the assumptions (5), and assume that*

$$\alpha > 1/2. \tag{34}$$

Moreover, assume that either $n \leq p + 2$, or $a(x, u, Du) \equiv a(x, Du)$. Then almost every boundary point $x \in \partial\Omega$, in the sense of the usual surface measure, is a regular point, i.e. the gradient is $C^{0,\alpha}$-regular in a neighborhood of x, relative to $\overline{\Omega}$. Moreover, when $p = 2$, we can allow $\alpha \geq 1/2 - \delta$, for some $\delta \equiv \delta(n, N, L/\nu) > 0$.

I explicitly remark that *the problem for the range $\alpha \in (0, 1/2)$ remains open*, and moreover it is not yet clear whether (34) is already optimal or not.

5 Measure data problems

The viewpoint and the techniques adopted in Sect. 2 reveal to be useful when considering *elliptic equations involving measure data*

$$\begin{cases} -\operatorname{div} a(x, Du) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{35}$$

Solutions are considered in the usual distributional sense [5]. Here I assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, μ is a Radon measure with finite total variation $|\mu|(\Omega) < \infty$, while $a\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathèodory vector field satisfying the following standard monotonicity, and Lipschitz assumptions:

$$\begin{cases} \nu(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_2 - z_1|^2 \le \langle a(x, z_2) - a(x, z_1), z_2 - z_1\rangle \\ |a(x, z_2) - a(x, z_1)| \le L(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}} |z_2 - z_1| \\ |a(x, 0)| \le Ls^{p-1}, \end{cases} \tag{36}$$

for any $z_1, z_2 \in \mathbb{R}^n$, $x \in \Omega$, where $p \in [2, n]$, $n \ge 2$, $0 < \nu \le L$, $s \ge 0$. At certain stage, I shall also require the following Lipschitz continuity assumption:

$$|a(x, z) - a(x_0, z)| \le L|x - x_0|(s^2 + |z|^2)^{\frac{p-1}{2}}, \quad \forall x, x_0 \in \Omega,\ z \in \mathbb{R}^n. \tag{37}$$

I shall now report on some recent, sharp results contained in [48] and [49]. A measure appearing in the right hand side in (35) is a natural source of singularities for solutions. The prime example is given by $\triangle_p u = \delta$, where δ is the Dirac measure charging the origin. In this case the unique solution is, up to a re-normalization constant, the "Green's function" $|x|^{\frac{p-n}{p-1}}$ when $p < n$, and $\log |x|$ for $p = n$. Nevertheless, Hölder continuity in the L^p-sense as in Sect. 2 still persists:

Theorem 12 *Under the assumptions* (36)–(37)*, there exists a solution* $u \in W_0^{1,1}(\Omega)$ *to the problem* (35) *such that*

$$Du \in W_{\text{loc}}^{\frac{1-\varepsilon}{p-1}, p-1}(\Omega, \mathbb{R}^n), \tag{38}$$

holds whenever $\varepsilon \in (0, 1)$*, and in particular*

$$Du \in W_{\text{loc}}^{1-\varepsilon,1}(\Omega, \mathbb{R}^n), \quad when \quad p = 2. \tag{39}$$

More in general

$$Du \in W_{\text{loc}}^{\frac{\sigma(q)-\varepsilon}{q}, q}(\Omega, \mathbb{R}^n), \tag{40}$$

whenever $\varepsilon \in (0, \sigma(q))$*, where*

$$p - 1 \le q < b, \quad \sigma(q) := n(1 - q/b), \quad b := \frac{n(p-1)}{n-1}. \tag{41}$$

The latter is the first higher regularity result for solutions to measure data problems: up to now it was only known the existence of a solution u such that

$$Du \in L^q, \quad \forall q < b = \frac{n(p-1)}{n-1}, \tag{42}$$

see [5], while in Theorem 12 higher derivatives are shown to exist; when $p = 2$ second derivatives "almost exist". Now consider the model case $p = 2$, and $\triangle u = f$. The standard Calderón-Zygmund theory asserts that $f \in L^{1+\varepsilon}$ implies $Du \in W^{1,1+\varepsilon}$ for every $\varepsilon > 0$, while the result does not hold in general for $\varepsilon = 0$. On the other hand (39) tells us that for non-linear elliptic problems with measure data the *Calderón-Zygmund theory continues below the borderline case* $W^{1,1}$, which is revealed to be a "break-point" rather than an "end-point". Inclusion (38), where we do not approach full first derivatives as $\varepsilon \searrow 0$, is the natural analogue of what already happens in the homogeneous case: $W^{1,p}$-solutions to $\triangle_p u = 0$ are not known, and by some people not expected to be twice differentiable; nevertheless their gradients are in suitable fractional Sobolev spaces. Inclusions (38)–(39) are a particular case of (40), which is in turn sharp for every choice of the parameters in (41). Indeed $Du \notin W^{\sigma(q)/q,q}_{\text{loc}}$ in general, otherwise applying Sobolev embedding in the fractional case we would have

$$Du \in W^{\sigma(q)/q,q}_{\text{loc}} \Longrightarrow Du \in L^{\frac{nq}{n-\sigma(q)}}_{\text{loc}},$$

but this is impossible since $nq/(n-\sigma(q)) = n(p-1)/(n-1) = b$, while $D|x|^{\frac{p-n}{p-1}} \notin L^b_{\text{loc}}$. On the other hand the application of the same embedding and (38) allow to locally recover the known integrability result for the gradient in (42).

The results proposed up to now are valid for general measures, and their sharpness stems from considering Dirac measures as just seen. It is therefore natural to wonder whether they change when considering measures diffusing on sets with higher Hausdorff dimension. A natural way to quantify this is to consider the following decay condition:

$$|\mu|(B_R) \leq cR^{n-\theta} \quad 0 \leq \theta \in [0, n], \tag{43}$$

for any ball B_R of radius R. Assuming (43) does not allow μ to concentrate on sets with Hausdorff dimension less than $n-\theta$, and improves the regularity of solutions in that the number θ replaces everywhere the dimension n. We recall that a measurable function w is the in weak space $\mathcal{M}^t(\Omega)$, $t \geq 1$, iff

$$\sup_{\lambda \geq 0} \lambda^t |\{x \in \Omega \,:\, |w| > \lambda\}| =: \|w\|^t_{\mathcal{M}^t(\Omega)} < \infty. \tag{44}$$

Moreover, w is in the weak Morrey space $\mathcal{M}^{t,\theta}(\Omega)$ iff $\|w\|_{\mathcal{M}^t(B_R)} \leq cR^{n-\theta}$ whenever $B_R \subset \Omega$, for some $c > 0$. Of course for $\theta < n$ it is $\mathcal{M}^{t,\theta} \subset \mathcal{M}^{t,n} \equiv \mathcal{M}^t$. We have

Theorem 13 *Under the assumptions* (36) *and* (43) *with $\theta \geq p$, there exists a solution $u \in W^{1,1}_0(\Omega)$ to the problem* (35) *such that*

$$Du \in \mathcal{M}^{m,\theta}_{\text{loc}}(\Omega, \mathbb{R}^n), \quad m := \frac{\theta(p-1)}{\theta - 1}. \tag{45}$$

In particular, in the limit case $\theta = p$ we have $Du \in \mathcal{M}^{p,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$.

Note that general measures satisfy (43) with $\theta = n$, when m reduces to b in (42), and we find back the known result $Du \in \mathcal{M}^b$, otherwise when $\theta < n$ when $m > b$. On the other hand, it is possible to prove that if μ satisfies (43) for $\theta < p$, then μ belongs to the dual of $W^{1,p}$ and problem (35) admits a solution $Du \in L^p$. The result $Du \in \mathcal{M}^p$ for $\theta = p$ perfectly reflects this fact. The fractional derivatives of the gradient are themselves in the natural Morrey space: referring to (38) and assuming (37) we have

$$\int_{B_R}\int_{B_R} \frac{|Du(x) - Du(y)|^{p-1}}{|x-y|^{n+1-\varepsilon}}\, dx\, dy \leq cR^{n-\theta}, \quad \forall \varepsilon \in (0,1). \tag{46}$$

See [48] for more details. Let me just recall that inequality (46) extends the classical results for $\triangle u = f$, where $f \in L^{q,\theta}$ implies $D^2 u \in L^{q,\theta}$ for $q > 1$; inequality (46) provides the natural analogue for $q = 1$. This is a regularity results in the so called Sobolev-Morrey spaces, introduced by Campanato [6,7].

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References

1. Acerbi, E., Fusco, N.: Semicontinuity problems in the calculus of variations. Arch. Ration. Mech. Anal. **86**, 125–145 (1984)
2. Acerbi, E., Fusco, N.: A regularity theorem for minimizers of quasiconvex integrals. Arch. Ration. Mech. Anal. **99**, 261–281 (1987)
3. Adams, D., Hedberg, L.I.: Function spaces and potential theory. Grundlehren der Mathematischen Wissenschaften 314. Springer-Verlag, Berlin (1996)
4. Ball, J.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Ration. Mech. Anal. **63**, 337–403 (1976/77)
5. Boccardo, L., Gallouët, T.: Nonlinear elliptic and parabolic equations involving measure data. J. Funct. Anal. **87**, 149–169 (1989)
6. Campanato, S.: Proprietà di inclusione per spazi di Morrey. Ricerche Mat. **12**, 67–86 (1963)
7. Campanato, S.: Proprietà di una famiglia di spazi funzionali. Ann. Scuola Norm. Sup. Pisa **18**(3), 137–160 (1964)
8. Campanato, S.: Equazioni ellittiche del II ordine e spazi L(2, λ). Ann. Mat. Pura Appl. **69**(4), 321–381 (1965)
9. Campanato, S., Cannarsa, P.: Differentiability and partial Hölder continuity of the solutions of nonlinear elliptic systems of order 2m with quadratic growth. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **8**(4), 285–309 (1981)
10. Campanato, S.: Hölder continuity of the solutions of some non-linear elliptic systems. Adv. Math. **48**, 16–43 (1983)
11. Carozza, M., Fusco, N., Mingione, G.: Partial regularity of minimizers of quasiconvex integrals with subquadratic growth. Ann. Mat. Pura Appl. **175**(4), 141–164 (1998)
12. Chipot, M., Evans, L.C.: Linearisation at infinity and Lipschitz estimates for certain problems in the calculus of variations. Proc. Roy. Soc. Edinburgh Sect. A **102**, 291–303 (1986)
13. Colombini, F.: Un teorema di regolarità alla frontiera per soluzioni di sistemi ellittici quasi lineari. Ann. Scuola Norm. Sup. Pisa **25**(3), 115–161 (1971)
14. Dacorogna, B.: Direct methods in the calculus of variations. Applied Mathematical Sciences, 78. Springer-Verlag, Berlin (1989)
15. De Giorgi, E.: Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. **125**(3), 25–43 (1957)
16. De Giorgi, E.: Un esempio di estremali discontinue per un problema variazionale di tipo ellittico. Boll. Un. Mat. Ital. **1**(4), 135–137 (1968)
17. Dorronsoro, J.R.: A characterization of potential spaces. Proc. Amer. Math. Soc. **95**, 21–31 (1985)
18. Duzaar, F., Grotowski, J.F.: Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation. Manuscripta Math. **103**, 267–298 (2000)
19. Duzaar, F., Kristensen, J., Mingione, G.: The existence of regular boundary points for non-linear elliptic systems. J. Reine Ang. Math. (Crelles J.) **602**, 17–58 (2007)
20. Duzaar, F., Steffen, K.: Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals. J. Reine Angew. Math. (Crelles J.) **546**, 73–138 (2002)
21. Evans, L.C.: Quasiconvexity and partial regularity in the calculus of variations. Arch. Ration. Mech. Anal. **95**, 227–252 (1986)
22. Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. Annals of Mathematics Studies, 105. Princeton University Press, Princeton, NJ (1983)
23. Giaquinta, M.: Direct methods for regularity in the calculus of variations. Nonlinear partial differential equations and their applications. Collège de France seminar, vol. VI (Paris, 1982/1983), 258–274, Res. Notes in Math., 109, Pitman, Boston, MA (1984)
24. Giaquinta, M.: The problem of the regularity of minimizers. In: Proceedings of the International Congress of Mathematicians, vol. 1, 2 (Berkeley, Calif.), pp. 1072–1083, Amer. Math. Soc., Providence, RI (1987)
25. Giaquinta, M.: Introduction to regularity theory for nonlinear elliptic systems. Lectures in mathematics ETH Zürich. Birkhäuser Verlag, Basel (1993)

26. Giaquinta, M., Giusti, E.: On the regularity of the minima of variational integrals. Acta Math. **148**, 31–46 (1982)
27. Giaquinta, M., Giusti, E.: Differentiability of minima of nondifferentiable functionals. Invent. Math. **72**, 285–298 (1983)
28. Giaquinta, M., Modica, G.: Almost-everywhere regularity for solutions of nonlinear elliptic systems. Manuscripta Math **28**, 109–158 (1979)
29. Giuffré, S., Idone, G.: Global regularity for solutions to Dirichlet problem for discontinuous elliptic systems with nonlinearity q $>$ 1 and with natural growth. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **8**(8), 519–524 (2005)
30. Giusti, E.: Direct methods in the calculus of variations. World Scientific Publishing Co., Inc, River Edge, NJ (2003)
31. Grotowski, J.F.: Boundary regularity for nonlinear elliptic systems. Calc. Var. Partial Differential Equations **27**, 2491–2512 (2002)
32. Grotowski, J.F.: Boundary regularity for quasilinear elliptic systems. Comm. Partial Differential Equations **27**, 2491–2512, 353–388 (2002)
33. Hamburger, C.: Regularity of differential forms minimizing degenerate elliptic functionals. J. Reine Angew. Math. (Crelles J.) **431**, 7–64 (1992)
34. Hamburger, C.: Quasimonotonicity, regularity and duality for nonlinear systems of partial differential equations. Ann. Mat. Pura Appl. **169**(4), 321–354 (1995)
35. Hamburger, C.: Partial boundary regularity of solutions of nonlinear superelliptic systems. Boll. Unione. Mat. Ital. Sez. B Artic. Ric. Mat. **10**(1), 63–81 (2007)
36. Hildebrandt, S., Widman, K.O.: Some regularity results for quasilinear elliptic systems of second order. Math. Z. **142**, 67–86 (1975)
37. Ivert, P.A.: Regularitätsuntersuchungen von Lösungen elliptischer Systeme von quasilinearen Differentialgleichungen zweiter Ordnung. Manuscripta Math **30**, 53–88 (1979/80)
38. Ivert, P.A.: Partial regularity of vector valued functions minimizing variational integrals. Preprint Univ. Bonn (1982)
39. Iwaniec, T.: The Gehring lemma. Quasiconformal mappings and analysis (Ann Arbor, MI, 1995), 181–204, Springer, New York (1998)
40. Kohn, R.V., Strang, G.: Optimal design and relaxation of variational problems I, II, III. Comm. Pure Appl. Math. **39**, 113–137, 139–182, 353–377 (1986)
41. Kristensen, J.: On the non-locality of quasiconvexity. Ann. Inst. H. Poincaré Anal. Non Linéaire **16**, 1–13 (1999)
42. Kristensen, J., Mingione, G.: The singular set of ω-minima. Arch. Ration. Mech. Anal. **177**, 93–114 (2005)
43. Kristensen, J., Mingione, G.: The singular set of minima of integral functionals. Arch. Ration. Mech. Anal. **180**, 331–398 (2006)
44. Kristensen, J., Mingione, G.: The singular set of Lipschitzian minima of multiple integrals. Arch. Ration. Mech. Anal. **184**(2), 341–369 (2007)
45. Manfredi, J.J.: Regularity of the gradient for a class of nonlinear possibly degenerate elliptic equations. Ph.D. Thesis. University of Washington, St. Louis.
46. Mingione, G.: The singular set of solutions to non-differentiable elliptic systems. Arch. Ration. Mech. Anal. **166**, 287–301 (2003)
47. Mingione, G.: Bounds for the singular set of solutions to non linear elliptic systems. Calc. Var. Partial Differential Equations **18**, 373–400 (2003)
48. Mingione, G.: The Calderón-Zygmund theory for elliptic problems with measure data. Ann. Scu. Norm. Sup. Pisa Cl. Sci (Ser. 5) **6**, 195–261 (2007)
49. Mingione, G.: Calderón-Zygmund estimates for measure data problems. C. R. Math. Acad. Sci. Paris **344**(7), 437–442 (2007)
50. Morrey, C.B.: Quasi-convexity and the lower semicontinuity of multiple integrals. Pacific J. Math. **2**, 25–53 (1952)
51. Müller, S.: Variational models for microstructure and phase transitions. Calculus of variations and geometric evolution problems (Cetraro, 1996). Lecture Notes in Math., 1713, pp. 85–210. Springer (1999)
52. Müller, S., Šverák, V.: Convex integration for Lipschitz mappings and counterexamples to regularity. Ann. of Math. **157**(2), 715–742 (2003)
53. Nečas, J.: Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity Theory of nonlinear operators. In: Proceedings of Fourth International Summer School, Academy of Science (Berlin, 1975) pp. 197–206
54. Rivière, T.: Everywhere discontinuous harmonic maps into spheres. Acta Math. **175**, 197–226 (1995)

55. Runst, T., Sickel, W.: Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations. Walter de Gruyter & Co., Berlin (1996)
56. Schoen, R., Uhlenbeck, K.: A regularity theory for harmonic maps. J. Differential Geom. **17**, 307–336 (1983)
57. Šverák, V., Yan, X.: Non-Lipschitz minimizers of smooth uniformly convex variational integrals. Proc. Natl. Acad. Sci. USA **99/24**, 15269–15276 (2002)
58. Uhlenbeck, K.: Regularity for a class of non-linear elliptic systems. Acta Math. **138**, 219–240 (1977)